

Uniformisation of a once-punctured annulus

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Abstract. The universal cover or the covering group of a hyperbolic Riemann surface X is important but hard to express explicitly. It can be, however, detected by the uniformisation and a suitable description of X . Beardon proposed five different ways to describe twice-punctured disks using fundamental domain, hyperbolic length, collar and extremal length in 2012. We parameterize a once-punctured annulus A in terms of five parameter pairs and give explicit formulas about the hyperbolic structure and the complex structure of A . Several degenerating cases are also treated.

Key words. Uniformisation, hyperbolic metric, punctured annulus, collar, extremal length.

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1 Introduction

The Uniformisation Theorem implies that every Riemann surface X is conformal equivalent to the quotient space \mathbb{H}/G , where G is a torsion-free Fuchsian group acting on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \text{Im}z > 0\}$, if X is not conformally equivalent to the Riemann sphere $\hat{\mathbb{C}}$, the complex plane \mathbb{C} , the once-punctured complex plane $\mathbb{C} \setminus \{a\}$ or a complex torus. It is, however, difficult to find an explicit form of the holomorphic universal cover π or the covering group G , except for several special cases (see e.g. [7, 16]). For a twice-punctured unit disk, Hempel and Smith [8, 9, 10] considered the uniformisation problem and the hyperbolic metric, and Beardon [3] provided five parameters to characterize the twice-punctured disk via the hyperbolic structure and the complex structure of it. Nevanlinna [13, I.3, I.4] introduced a method to regard the puncture as the extremal case when a boundary curve shrinks to a single point. In this article we give five parameter pairs to uniformize a once-punctured annulus A . These parameter pairs can be divided into two classes which are corresponding to the hyperbolic structure and complex structure of A , respectively.

Let γ be a simple closed geodesic on a hyperbolic surface X , and let

$$C_\theta(\gamma) := \{x \in X : \delta_X(x, \gamma) < \sinh^{-1}(\tan \theta)/2\}, \quad (1.1)$$

where δ_X is the hyperbolic distance on X of the Gaussian curvature -1 . $C_\theta(\gamma)$ is called a *collar about γ of angular width θ* if it is doubly connected. When $X \setminus \gamma$ has a doubly connected component W and γ is homotopic to no puncture, γ is homotopic to a border of X , and γ is called *peripheral*. If $C_\theta(\gamma)$ is a collar, $\tilde{C}_\theta(\gamma) := C_\theta(\gamma) \cup W$ is a doubly connected subdomain

in X containing γ . We will refer to $\tilde{C}_\theta(\gamma)$ as a *peripheral collar about γ of angular width θ* . In Section 2 we will give the details of peripheral collars.

We will use Legendre's complete elliptic integrals $K(r)$ of the first kind. Let $K'(r) := K(r') = K(\sqrt{1-r^2})$ for $0 < r < 1$, and $\mu(r) := \pi K'(r)/(2K(r))$. The properties of $K(r)$, $K'(r)$ and $\mu(r)$ are given in Section 2.

After an application of a rotation and a similarity, we only need to consider the punctured annulus

$$A := \{z : 1/R < |z| < R\} \setminus \{a\}, \quad R > 1, \quad 1/R < a < R. \quad (1.2)$$

We denote $C_1 := \{z : |z| = 1/R\}$, $C_2 := \{z : |z| = R\}$, and let $\mathcal{C}_1, \mathcal{C}_2$ be the free homotopy classes of the circles $\{z : |z| = r_1\}$, $\{z : |z| = r_2\}$ in A , respectively, where $a < r_1 < R$, $1/R < r_2 < a$. So \mathcal{C}_1 separates $C_1 \cup \{a\}$ from C_2 , and \mathcal{C}_2 separates $C_2 \cup \{a\}$ from C_1 . Let γ_1, γ_2 be the hyperbolic geodesics in $\mathcal{C}_1, \mathcal{C}_2$. The main results in this article are as follows.

Theorem 1.1 *Let l_1, l_2 be the hyperbolic lengths of γ_1, γ_2 , and θ_1, θ_2 be the angular widths of the maximal peripheral collars about γ_1, γ_2 in the punctured annulus A . Then (l_1, l_2) and (θ_1, θ_2) satisfy*

$$\cos \theta_1 = \frac{\sinh \frac{l_1}{2}}{\cosh \frac{l_1}{2} + \cosh \frac{l_2}{2}}, \quad \cos \theta_2 = \frac{\sinh \frac{l_2}{2}}{\cosh \frac{l_1}{2} + \cosh \frac{l_2}{2}}. \quad (1.3)$$

Theorem 1.2 *Let λ_1 and λ_2 be the extremal lengths of \mathcal{C}_1 and \mathcal{C}_2 . Select a positive number q such that $\mu(q) = 4 \log R$ and let $\mathcal{K} := K(q)$, $\mathcal{K}' := K'(q)$. Then*

$$\lambda_1 = \frac{2\pi}{\mu(p_1)}, \quad \lambda_2 = \frac{2\pi}{\mu(p_2)}, \quad (1.4)$$

where

$$p_1 = \frac{\sqrt{q}(\operatorname{dn} u_1 + 1)}{q + \operatorname{dn} u_1}, \quad p_2 = \frac{\sqrt{q}(\operatorname{dn} u_2 + 1)}{q + \operatorname{dn} u_2} \quad (1.5)$$

with

$$u_1 = \frac{2\mathcal{K}}{\pi} \log Ra, \quad u_2 = \frac{2\mathcal{K}}{\pi} \log \frac{R}{a}, \quad (1.6)$$

and the Jacobian elliptic function dn in (1.5) has the modulus $q' = \sqrt{1-q^2}$.

Theorems 1.1 and 1.2 reveal the connections between (θ_1, θ_2) and (l_1, l_2) , (λ_1, λ_2) and (R, a) , respectively. This implies that we have two classes of parameter pairs to describe A , one class related to (l_1, l_2) and the other to (R, a) . Actually the two parameter classes are corresponding to two kinds of structures on A , hyperbolic and complex structures.

This article is organized in the following way. Section 2 is about preliminaries, and we construct the covering group G and the fundamental domain D_A of A on \mathbb{H} there. The construction of G gives rise to two real numbers r and k , which also form a parameter pair of A . Section 3 is devoted to the hyperbolic structure of A . We prove Theorem 1.1 using the parameter pair of the covering group G given in Section 3. Section 4 is devoted to the complex structure of A , and we prove Theorem 1.2 there. Section 5 is about the degenerating cases when the puncture is tending to one of the boundary circles, and when the boundaries are shrinking to points. We give an observation for the once-punctured annulus A when only one boundary circle of A is shrinking to a single point, in which case A is becoming a twice-punctured disk.

2 Preliminary

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The hyperbolic metrics on \mathbb{H} and \mathbb{D} of the Gaussian curvature -1 are given by

$$\rho_{\mathbb{H}}(z)|dz| = \frac{|dz|}{\operatorname{Im} z}, \quad \rho_{\mathbb{D}}(z)|dz| = \frac{2|dz|}{1 - |z|^2}. \quad (2.7)$$

They induce the hyperbolic lengths $\delta_{\mathbb{H}}(x, y)$ and $\delta_{\mathbb{D}}(x, y)$ between two points (or sets) x and y in \mathbb{H} and \mathbb{D} . The universal covering space \tilde{X} of a Riemann surface X can be tessellated by a fundamental domain and its images under the covering group G acting on \tilde{X} . For a hyperbolic surface X , we take \tilde{X} to be the upper half plane \mathbb{H} or the open unit disk \mathbb{D} . A domain $D \subseteq \tilde{X}$ is called a *fundamental domain* for G if D satisfies the following two conditions: $g(D) \cap D = \emptyset$ for all $g \in G$, $g \neq \operatorname{Id}$; $\bigcup_{g \in G} g(D) = \tilde{X}$. The hyperbolic metric on \tilde{X} can be projected under the quotient mapping to a metric on X which is called the *hyperbolic metric on X* . It is the intrinsic metric on X . It is independent of the choice of the universal cover from \tilde{X} .

We identify the Möbius transformation

$$\phi(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C}, \quad (2.8)$$

with the 2×2 complex matrices $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{PSL}(2, \mathbb{C})$ which is also denoted by ϕ , and define the *trace* of ϕ by $\operatorname{tr} \phi = \pm(a + d)$, such that $\operatorname{tr}^2 \phi = (a + d)^2$ is well defined. Möbius transformations preserve the hyperbolic metric. All the conformal isometries of \mathbb{H} are Möbius transformations with a, b, c, d being real numbers. The *translation length* of ϕ is defined by $T(\phi) = \inf_{z \in \mathbb{H}} \delta_{\mathbb{H}}(z, \phi(z))$, where $\delta_{\mathbb{H}}$ is the hyperbolic distance on \mathbb{H} . When ϕ is hyperbolic, $T(\phi) > 0$. It is known that $2 \cosh(T(\phi)/2) = |\operatorname{tr} \phi|$ (see [2, 7.34]).

For a simple closed geodesic γ of hyperbolic length l in a hyperbolic surface X , let $C_{\theta}(\gamma)$ be a collar about γ of width θ , $0 < \theta < \frac{\pi}{2}$. Then there exists a hyperbolic transformation f in the covering group G of X , such that $f(C_{\theta}(\gamma)) = C_{\theta}(\gamma)$ and $\operatorname{Stab}_G(C_{\theta}(\gamma)) = \langle f \rangle$, where $\operatorname{Stab}_G(C_{\theta}(\gamma))$ is the stabilizer of $C_{\theta}(\gamma)$ in G . Moreover, $C_{\theta}(\gamma)$ satisfies $h(C_{\theta}(\gamma)) \cap C_{\theta}(\gamma) = \emptyset$ if $h \in G \setminus \langle f \rangle$. By conjugation we may assume that $f : z \mapsto k^2 z$, $k = \exp \frac{l}{2}$, then $\{z : 1 < |z| < k^2, \frac{\pi}{2} - \theta < \arg z < \frac{\pi}{2} + \theta\}$ is a fundamental domain for $\langle f \rangle$ in $C_{\theta}(\gamma)$. The collar lemma showed that if θ satisfies

$$\tan \theta \leq \frac{2}{k - k^{-1}} \quad (2.9)$$

C_{θ} is a collar about γ of width θ (see [11, Lemma 7.7.1]). In addition, if γ is peripheral, $\{z : 1 < |z| < k^2, \frac{\pi}{2} - \theta < \arg z < \pi\}$ (or $\{z : 1 < |z| < k^2, 0 < \arg z < \pi - \theta\}$ by conjugation) is a fundamental domain for $\langle f \rangle$ in the peripheral collar $\tilde{C}_{\theta}(\gamma)$ which contains $C_{\theta}(\gamma)$.

We have the following lemma for the covering group G of A .

Lemma 2.1 *For the punctured annulus A given by (1.2), there exist two real numbers k and r , $1 < r < k$, such that the group G generated by the hyperbolic transformation f and the parabolic transformation g is the covering group of A acting on \mathbb{H} , where*

$$f = \pm \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}, \quad g = \frac{\pm 1}{r - 1} \begin{pmatrix} 2r & -(r + 1) \\ r + 1 & -2 \end{pmatrix}. \quad (2.10)$$

Proof. We first construct a fundamental domain of the covering group acting on \mathbb{D} and a universal cover $\pi : \mathbb{D} \rightarrow A$ of the punctured annulus A . A can be divided into two pieces, the upper half A^+ and the lower half A^- , by the three Euclidean line segments $\eta_1 := (-R, -1/R)$, $\eta_2 := (1/R, a)$, $\eta_3 := (a, R)$ lying on the real axis. Since A is symmetric with respect to the real axis, each lift of η_i under π is a hyperbolic line in \mathbb{D} for $i = 1, 2, 3$ (see Hemple and Smith [10, Section 2]). We take one component $D_{A^+}^0$ of the pre-image of A^+ under π , then the inverse of π has a univalent branch $\tau : A^+ \rightarrow D_{A^+}^0$. We now describe $\partial D_{A^+}^0 \cap \mathbb{D}$. The function τ can be extended continuously to $\overline{A^+} \cap A = A^+ \cup \eta_1 \cup \eta_2 \cup \eta_3$. We denote the extension of τ still by τ , so that $\tau(\eta_1) \cup \tau(\eta_2) \cup \tau(\eta_3) = \partial D_{A^+}^0 \cap \mathbb{D}$. Since $\tau(\eta_i)$ is a hyperbolic line in \mathbb{D} for $i = 1, 2, 3$, by conjugacy of Möbius transformations if necessary, we may assume that $\tau(\eta_3)$ is the diameter $(-1, 1)$ of \mathbb{D} , with $-1 = \tau(a)$ and $1 = \tau(R)$. Then $\tau(\eta_2)$ and $\tau(\eta_3)$ are hyperbolic lines in \mathbb{D} , where $\tau(\eta_2)$ and $\tau(\eta_3)$ are parallel with the common endpoint -1 , and $\tau(\eta_1)$ is disjoint with $\tau(\eta_2)$ or $\tau(\eta_3)$, in accordance with the position of η_1, η_2 and η_3 in A^+ . In such a way the upper half punctured annulus A^+ is mapped onto the region $D_{A^+}^0$ bounded by $\tau(\eta_i)$, $i = 1, 2, 3$, and the two arcs on $\partial \mathbb{D}$ joining 1 and $\tau(-R)$, $\tau(-1/R)$ and $\tau(1/R)$, shown in Figure 1(a).

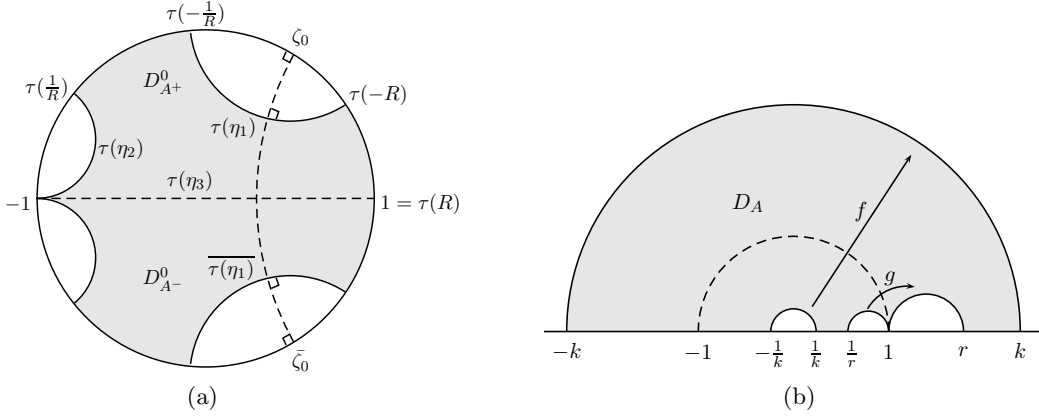


Figure 1.

Denote $A^* := A \setminus (\eta_1 \cup \eta_2)$. Since A^* is symmetric with respect to η_3 , so $\tau(A^*)$ is symmetric with respect to $\tau(\eta_3) = (-1, 1)$. We reflect $D_{A^+}^0$ along $(-1, 1)$ to obtain $D_{A^-}^0$, then $D_{A^+}^0 \cup D_{A^-}^0 \cup (-1, 1) = \tau(A^*) =: D_A^0$ is a fundamental domain of A in \mathbb{D} . Side $\tau(\eta_1)$ is paired with its conjugate $\overline{\tau(\eta_1)}$ by a hyperbolic transformation \tilde{f} fixing two points, say, ζ_0 and $\bar{\zeta}_0$, and $\tau(\eta_2)$ is paired $\tau(\eta_2)$ by a parabolic transformation \tilde{g} fixing -1 . By a Möbius transformation, under conjugacy, $\varphi : \mathbb{D} \rightarrow \mathbb{H}$ satisfying $\varphi(-1) = 1$, $\varphi(1) = -1$ and $\varphi(\zeta_0) = 0$, $\varphi(\bar{\zeta}_0) = \infty$, $f := \varphi \tilde{f}$, $g := \varphi \tilde{g}$ have the form (2.10) with a parameter pair (k, r) , $1 < r < k$, and $G = \langle f, g \rangle$ is the covering group acting on \mathbb{H} . Moreover, we let $S_1 := \{z \in \mathbb{H} : |z - \frac{r+1}{2r}| = \frac{r-1}{2r}\}$, $S_2 := \{z \in \mathbb{H} : |z| = \frac{1}{k}\}$, and let D_A be the subdomain of \mathbb{H} bounded by S_1 , $g(S_1)$, S_2 , $f(S_2)$ and three Euclidean line segments $(-k, -k^{-1})$, (k^{-1}, r^{-1}) , (r, k) . Then D_A^0 is mapped onto the fundamental domain D_A of G in \mathbb{H} , shown in Figure 1(b). By the symmetry of A we know γ_1 is orthogonal to η_1 , γ_1 is a lift of the hyperbolic line with two endpoints ζ_0 and $\bar{\zeta}_0$ in \mathbb{D} under π , which means that γ_1 is corresponding to \tilde{f} acting on \mathbb{D} , thus f on \mathbb{H} . \square

The concept of the extremal length can be established as follows. Let $\Omega \subseteq \mathbb{C}$ and Γ be a collection of finite unions of curves in Ω . All of the metrics which are conformal with respect to the Euclidean metric can be defined in terms of a density $\varrho(z)|dz|$ where $\varrho(z)$ is a non-negative Borel measurable function on Ω . Then the length of $\gamma \in \Gamma$ and the area of Ω with

respect to $\varrho(z)$ are given by

$$L(\gamma, \varrho) = \int_{\gamma} \varrho(z) |dz|, \quad A(\Omega, \varrho) = \int_{\Omega} \varrho(z)^2 dx dy$$

with $z = x + iy$. These two quantities do not change under conformal mappings. We let $L(\Gamma, \varrho) = \inf_{\gamma \in \Gamma} L(\gamma, \varrho)$. The *extremal length* of Γ in Ω is defined by

$$\lambda(\Gamma) = \sup_{\varrho} \frac{L(\Gamma, \varrho)^2}{A(\Omega, \varrho)}, \quad (2.11)$$

where the supremum is taken over all conformal densities such that $0 < A(\Omega, \varrho) < \infty$. The extremal length is a conformal invariant and does not change when ϱ is multiplied by a constant (see [11, 7.6.2]).

Let

$$K(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}$$

with $0 < r < 1$ be Legendre's complete elliptic integrals of the first kind, and

$$\operatorname{sn}(u, r) = \tau \quad \text{where} \quad u = \int_0^{\tau} \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}$$

be the Jacobian elliptic sine function. Function $\operatorname{sn}(u, r)$ is a bijection from $[-K(r), K(r)]$ onto $[-1, 1]$ (see [1]). Two other functions are then defined by

$$\operatorname{cn}(u, r) = \sqrt{1 - \operatorname{sn}^2(u, r)}, \quad \operatorname{dn}(u, r) = \sqrt{1 - r^2 \operatorname{sn}^2(u, r)}. \quad (2.12)$$

The three functions $\operatorname{sn}(u, r)$, $\operatorname{cn}(u, r)$ and $\operatorname{dn}(u, r)$ are called *Jacobian elliptic functions* (see [5] for the fundamental relations and addition formulas of them). The parameter $r \in (0, 1)$ is called the *modulus* and the *complementary modulus* of r is $r' = \sqrt{1 - r^2}$. If the modulus is fixed we can write Jacobian elliptic functions as $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$ for short. In the rest of this article we use $K(r)$ to refer to the Legendre's complete elliptic integral of the first kind and denote $K'(r) = K(r') = K(\sqrt{1 - r^2})$. We define the *normalized quotient function*

$$\mu(r) = \frac{\pi K'(r)}{2 K(r)}$$

for $0 < r < 1$, then $\mu(r)$ is a strictly decreasing homeomorphism of the interval $(0, 1)$ onto $(0, \infty)$ with limit values $\mu(0+) = \infty$, $\mu(1-) = 0$ (see [1]).

3 Hyperbolic structure of A

To prove Theorem 1.1 we need the following two theorems about the connections between (θ_1, θ_2) and (k, r) , (l_1, l_2) and (k, r) .

Theorem 3.1 *Suppose that θ_1 and θ_2 are the angular widths of the maximal peripheral collars about γ_1, γ_2 . Then we have*

$$\cos \theta_1 = \frac{r-1}{r+1}, \quad \cos \theta_2 = \frac{t-1}{t+1} = \frac{2r(r+1) - 2\delta}{\delta(r+1) - (r+1)^2}, \quad (3.13)$$

where

$$t = \frac{(r-1)(r+1+\delta)}{(r+3)\delta - (r+1)(3r+1)}, \quad \delta = k^2 + r - \sqrt{(k^2-1)(k^2-r^2)} \quad (3.14)$$

with k and r as in (2.10).

Proof. In the fundamental domain D_A shown in Figure 1(b), we take $S_1 = \{z \in \mathbb{H} : |z - \frac{r+1}{2r}| = \frac{r-1}{2r}\}$. It is clear that the common tangent Euclidean line of S_1 and $g(S_1)$ is going through the origin. Denote the segment of the common tangent line in \mathbb{H} by L and the tangent point where L and S_1 are tangent by P_1 , the tangent point where L and $g(S_1)$ are tangent by P_2 . We assert that the maximal peripheral collar about the axis of f is the region C bounded by the non-positive real axis and L in \mathbb{H} . It can be seen as follows. From the foundation of elementary geometry we know that the center of the Euclidean circle \mathcal{C}_0 passing through P_1 , P_2 and 1 is in L , and \mathcal{C}_0 is tangent to the real axis at 1. Then \mathcal{C}_0 is a horocycle orthogonal to S_1 and $g(S_1)$, and it is invariant under g . Hence $P_2 = g(P_1)$, which means $g(S_1)$ is tangent to $g(L)$ at P_2 if we note that g is one-to-one on \mathbb{H} , and then L is tangent to $g(L)$ at P_2 , where $g(L)$ is the hypercircle in the exterior of D_A with two endpoints in the interval $(1, r)$. This verifies the maximality of the collar C . To describe C we consider the angular width θ between L and the positive imaginary axis.

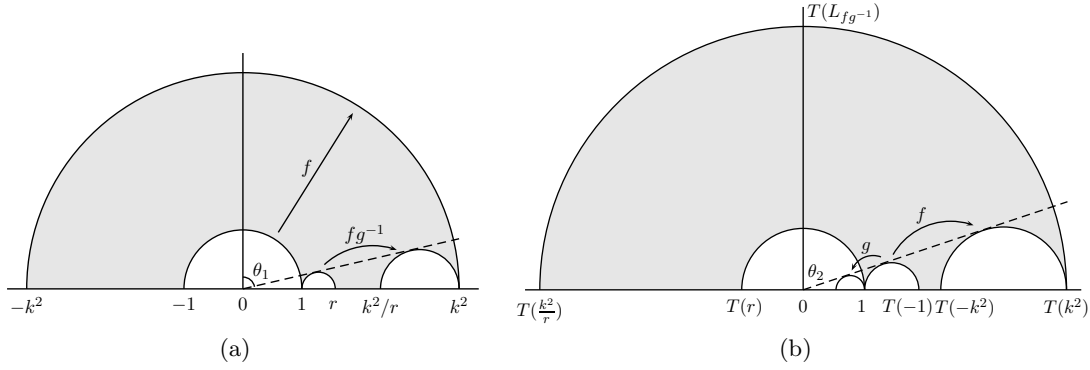


Figure 2.

For f and g given by (2.10), we have

$$fg^{-1} = \frac{\pm 1}{r-1} \begin{pmatrix} 2k & -k(r+1) \\ (r+1)k^{-1} & -2rk^{-1} \end{pmatrix}, \quad (3.15)$$

which is a hyperbolic transformation. Since γ_1, γ_2 are corresponding to f, fg^{-1} , θ_1 and θ_2 are angular widths of the maximal peripheral collars about the axes of f, fg^{-1} , respectively. θ_1 is shown in Figure 2(a), thus $\cos \theta_1 = \frac{r-1}{r+1}$. Now we consider θ_2 , which is shown as in Figure 3, where x_1 and x_2 are the fixed points of fg^{-1} . By some Möbius transformation T with

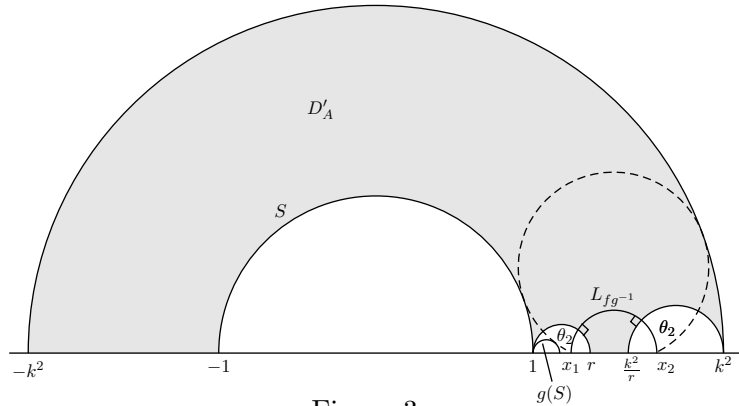


Figure 3.

$T(x_1) = 0, T(x_2) = \infty$, D'_A in Figure 3 can be mapped onto the fundamental domain shown in Figure 2(b) with the vertices $T(-k^2), T(-1), T(1), T(r), T(x_2), T(k^2)$ and the angle θ_2 .

To identify T , without loss of generality, we may assume that $T(1) = 1$. Then after some computation, we have

$$x_1 = \frac{\delta}{r+1} \in \left(\frac{r}{2} + \frac{1}{2}, r \right), \quad x_2 = \frac{k^2}{x_1}.$$

So

$$T(z) = \frac{(1-r)z - x_1(1-r)}{(r+1-2x_1)z + x_1(r+1) - 2r}.$$

Thus

$$T(-1) = \frac{(r-1)(r+1+\delta)}{(r+3)\delta - (r+1)(3r+1)} := t.$$

We have $t > 1$ provided that $2r < \frac{1}{2}(r+1)^2 < \delta < r(r+1)$. That is shown in Figure 2(b) and then $\cos \theta_2 = \frac{t-1}{t+1}$. \square

Remark 1. Theorem 3.1 is an improvement of the collar lemma. The maximal collar defined by (2.9) with equality is smaller than the collar decided by θ_1 and θ_2 in (3.13). We denote the angular widths of the collars defined by (2.9) about the axes of f and fg^{-1} by θ'_1 and θ'_2 . Then $\cos \theta'_1 = \frac{k^2-1}{k^2+1}$ by (2.9). Since the function $\frac{x-1}{x+1}$ is monotonically increasing for $x \in [1, \infty)$, and $1 < r < k < k^2$, then $\theta'_1 < \theta_1$ from the expression of $\cos \theta_1$ in (3.13). From the symmetry given by (1.3), or by pure computation, we have $\theta'_2 < \theta_2$ for fg^{-1} .

Lemma 3.2 *In the punctured annulus A , for the lengths l_1, l_2 of γ_1, γ_2 , and (k, r) as in (2.10), we have*

$$2 \cosh \frac{l_1}{2} = k + \frac{1}{k}, \quad 2 \cosh \frac{l_2}{2} = \frac{2}{r-1} \left(k - \frac{r}{k} \right). \quad (3.16)$$

Proof. Since γ_1, γ_2 are corresponding to f, fg^{-1} , respectively, l_1 and l_2 are the translation lengths of f and fg^{-1} . Thus (3.16) is obvious from (3.15). \square

Proof of Theorem 1.1. At first we solve that

$$r = \frac{k \cosh \frac{l_2}{2} + k^2}{k \cosh \frac{l_2}{2} + 1}$$

from the second equation of (3.16), then by (3.13) and the first equation of (3.16),

$$\cos \theta_1 = \frac{r-1}{r+1} = \frac{k - \frac{1}{k}}{2 \cosh \frac{l_2}{2} + k + \frac{1}{k}} = \frac{\sinh \frac{l_1}{2}}{\cosh \frac{l_1}{2} + \cosh \frac{l_2}{2}}$$

as required. The expression for $\cos \theta_2$ in (1.3) can be obtained by symmetry. \square

Remark 2. We compare θ_1 and θ_2 , l_1 and l_2 . By (1.3) and

$$t - r = \frac{(r+1)^2(3r-1-\delta)}{(r+3)\delta - (r+1)(3r+1)}, \quad -(r-1)^2 < 3r-1-\delta < \frac{1}{2}(r-1)(r-3),$$

we have the following inequalities. When $1 < r < 3$,

$$\begin{aligned} t > r, \quad \theta_1 > \theta_2 \text{ and } l_1 < l_2, & \quad \text{if } \sqrt{\frac{3r-1}{3-r}} < k, \\ t = r, \quad \theta_1 = \theta_2 \text{ and } l_1 = l_2, & \quad \text{if } \sqrt{\frac{3r-1}{3-r}} = k, \end{aligned}$$

$$t < r, \theta_1 < \theta_2 \text{ and } l_1 > l_2, \quad \text{if } k < \sqrt{\frac{3r-1}{3-r}};$$

when $r \geq 3$, $t < r$, $\theta_1 < \theta_2$ and $l_1 > l_2$. Then the following corollary is obtained provided that $a = 1$ is the midpoint between $1/R$ and R in the hyperbolic metric on A .

Corollary 3.3 *In the punctured annulus $A = \{z : 1/R < |z| < R\} \setminus \{1\}$, the covering group G of A is generated by*

$$f(z) = \frac{3r-1}{3-r}z, \quad g(z) = \frac{2rz - (r+1)}{(r+1)z - 2},$$

where $1 < r < 3$ and r is related to R in some unknown way.

Remark 3. Here we provide another way to prove Theorem 1.1 by the use of the pants decomposition for a hyperbolic manifold with a cusp. It is different from the description presented in this article, but it is meaningful for the discussion of pants partitions and spectral questions. We only show the rough idea here without rigorous proof. For more details on pairs of pants and collars (see [4]). Note that γ_1 and γ_2 are orthogonal to the real axis, at points, say, s_1 and s'_1 , s_2 and s'_2 , respectively. Let $\tilde{C}_{\theta_1}(\gamma_1)$ be the maximal peripheral collar about geodesic γ_1 of angular width θ_1 . Then, by symmetry, the inner boundary l of $\tilde{C}_{\theta_1}(\gamma_1)$ is orthogonal to segments $(-R, -1/R)$, (a, R) , and tangent to itself at a point between $1/R$ and a . The sketch of l , γ_1 and γ_2 is shown in Figure 4. The punctured domain bounded by

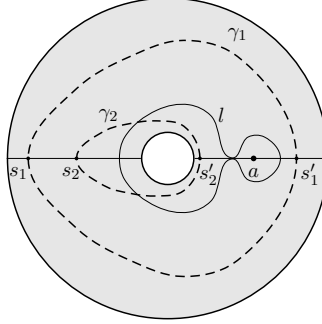


Figure 4.

γ_1 and γ_2 with puncture a is conformally equivalent to a pair of pants with a cusp, called a Y-piece with a cusp (see [4]), shown in Figure 5(a). We use the same notations as in A . Then, in the Y-piece, l is a curve orthogonal to geodesic ray (a, s'_1) and tangent to (a, s'_2) , and also orthogonal to the geodesic line containing (s_1, s_2) . It is known that the Y-piece in Figure 5(a) with a cusp can be decomposed into two isometric pentagons with four right angles, one of which is shown in Figure 5(b), where l is corresponding to a hypercycle, still denoted by l , tangent to (a, s_2) . It is easy to see that, in this pentagon, the angle between l and the geodesic containing side $\frac{1}{2}\gamma_1$ is the angular width θ_1 of $\tilde{C}_{\theta_1}(\gamma_1)$. So that we can obtain (1.3) for θ_1 only from properties of a right-angled pentagon, and then for θ_2 by symmetry. Moreover, the hyperbolic length of segment (s_1, s_2) in Figure 5(b) can be obtained,

$$\sinh l(s_1, s_2) = \frac{\cosh \frac{l_1}{2} + \cosh \frac{l_2}{2}}{\sinh \frac{l_1}{2} \sinh \frac{l_2}{2}},$$

it is the distance between γ_1 and γ_2 in Figure 4.

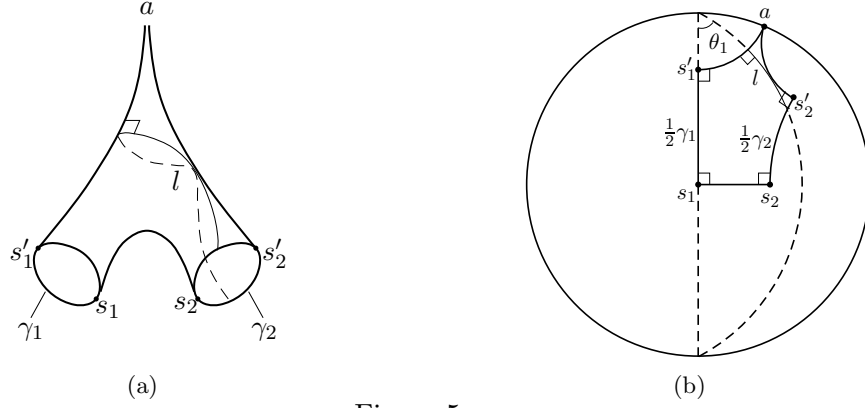


Figure 5.

4 Complex structure of A

For the proof of Theorem 1.2 we need the following three lemmas.

Lemma 4.1 (see 6.26, 6.27 in [1]) *For $0 < q < 1$ let $\mathcal{K} := K(q)$, $\mathcal{K}' := K'(q)$ and select $b = \exp(-\pi\mathcal{K}'/(4\mathcal{K}))$. Then the conformal mapping ω defined by*

$$\omega(z) = \sqrt{q} \operatorname{sn} \left(\frac{2i\mathcal{K}}{\pi} \operatorname{Log} \frac{z}{b} + \mathcal{K}, q \right)$$

is unique up to rotations and takes the annulus $b < |z| < 1$ onto the unit disk $|\omega(z)| < 1$ minus the slit $[-\sqrt{q}, \sqrt{q}]$, where Log is the principal branch of the logarithm.

Lemma 4.2 *For given numbers q and α , $0 < \sqrt{q} < \alpha < 1$, the unique Möbius transformation σ which preserves \mathbb{D} and satisfies $\sigma(-1) = -1$, $\sigma(1) = 1$, $\sigma(-\sqrt{q}) = 0$ is given by*

$$\sigma(z) = \frac{z + \sqrt{q}}{\sqrt{q}z + 1}.$$

Proof. Note that the correspondences between three distinct points and their images on $\hat{\mathbb{C}}$ decide a Möbius transformation and a Möbius transformation preserving the unit disk has the form $\pm \begin{pmatrix} b_1 & \overline{b_2} \\ b_2 & \overline{b_1} \end{pmatrix} \in \operatorname{PSL}(2, \mathbb{C})$, then the conclusion is obvious. \square

The actions of ω and σ are shown in Figure 6.

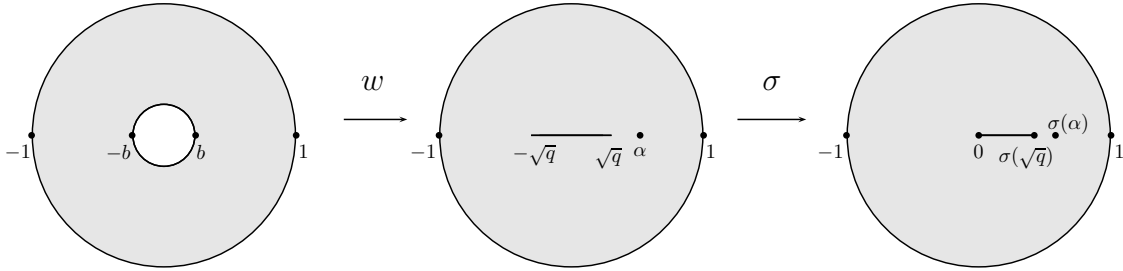


Figure 6.

Lemma 4.3 *Let $\tilde{\mathcal{C}}$ be the family of loops in \mathbb{D} separating 0 and x from the unit circle $\partial\mathbb{D}$, $0 < x < 1$, and \mathcal{C} be the family of loops in \mathbb{D} separating the slit $(0, x)$ from $\partial\mathbb{D}$. Then the extremal lengths of $\tilde{\mathcal{C}}$ and \mathcal{C} are*

$$\lambda(\tilde{\mathcal{C}}) = \lambda(\mathcal{C}) = \frac{2\pi}{\mu(x)}. \quad (4.17)$$

Lemma 4.3 can be found in [12, 2.1] for the extremal length of family \mathcal{C} as the reciprocal of the modulus of Grötzsch's extremal domain. On the twice-punctured unit disk $\mathbb{D} \setminus \{\pm r\}$ Ohtsuka [15, Theorem 2.56] gave the extremal length of the curve family which separates punctures $\{-r, r\}$ from the unit circle. If we note that there exists a Möbius transformation which maps Ohtsuka's domain $\mathbb{D} \setminus \{\pm r\}$ to the domain $\mathbb{D} \setminus \{0, x\}$ and $\mu(\frac{2r}{r^2+1}) = \frac{1}{2}\mu(r^2)$, we can obtain the extremal length of $\tilde{\mathcal{C}}$ in (4.17).

Proof of Theorem 1.2. At first, we take the mappings

$$\varphi_1(z) = \frac{z}{R}, \quad \varphi_2(z) = \frac{1}{Rz},$$

then $\varphi_1(A)$, $\varphi_2(A)$ are still punctured annuli

$$\varphi_1(A) = \left\{ z : \frac{1}{R^2} < |z| < 1 \right\} \setminus \left\{ \frac{a}{R} \right\}, \quad \varphi_2(A) = \left\{ z : \frac{1}{R^2} < |z| < 1 \right\} \setminus \left\{ \frac{1}{aR} \right\}.$$

Next let $b = 1/R^2$ and define

$$\omega(z) = \sqrt{q} \operatorname{sn} \left(\frac{2i\mathcal{K}}{\pi} \operatorname{Log} R^2 z + \mathcal{K}, q \right),$$

where q satisfies $\mu(q) = 4 \log R$. Then from Lemma 4.1, $\omega(z)$ maps $\varphi_1(A)$ and $\varphi_2(A)$ onto two punctured slit disks $\mathbb{D} \setminus ([-\sqrt{q}, \sqrt{q}] \cup \{\alpha_1\})$ and $\mathbb{D} \setminus ([-\sqrt{q}, \sqrt{q}] \cup \{\alpha_2\})$, respectively, where

$$\alpha_1 = \omega \left(\frac{a}{R} \right) = \sqrt{q} \operatorname{sn} (iu_1 + \mathcal{K}, q), \quad \alpha_2 = \omega \left(\frac{1}{aR} \right) = \sqrt{q} \operatorname{sn} (iu_2 + \mathcal{K}, q)$$

with u_1 and u_2 given by (1.6). We note that

$$\operatorname{sn} (iu_j + \mathcal{K}, q) = \frac{\operatorname{cn}(iu_j, q)}{\operatorname{dn}(iu_j, q)},$$

and

$$\operatorname{cn}(iu_j, q) = \frac{1}{\operatorname{cn}(u_j, q')}, \quad \operatorname{dn}(iu_j, q) = \frac{\operatorname{dn}(u_j, q')}{\operatorname{cn}(u_j, q')}$$

for $j = 1, 2$ (see (120.02), (122.03) and (125.02) in [5]). Then $\alpha_1 = \frac{\sqrt{q}}{\operatorname{dn} u_1}$, $\alpha_2 = \frac{\sqrt{q}}{\operatorname{dn} u_2}$ with the modulus of dn being $q' = \sqrt{1 - q^2}$. So for $j = 1, 2$,

$$p_j := \sigma(\alpha_j) = \frac{\alpha_j + \sqrt{q}}{\sqrt{q}\alpha_j + 1} = \frac{\sqrt{q}(\operatorname{dn} u_j + 1)}{q + \operatorname{dn} u_j}.$$

For $\tilde{\mathcal{C}}$ and \mathcal{C} defined in Theorem 1.2, since $\tilde{\mathcal{C}} \supset \mathcal{C}_j \supset \mathcal{C}$, then $\lambda(\tilde{\mathcal{C}}) \leq \lambda(\mathcal{C}_j) \leq \lambda(\mathcal{C})$, $j = 1, 2$. Thus (1.4) is obtained from (4.17). \square

Corollary 4.4 *In the same assumption as Theorem 1.2, when the puncture $a = 1$,*

$$\lambda_1 = \lambda_2 = \frac{2\pi}{\mu(p)},$$

where

$$p = \frac{\sqrt{q} \left(\sqrt{\sqrt{q}(\sqrt{1+q}+1)} + \sqrt{\sqrt{1+q}+\sqrt{q}} \right)}{q\sqrt{\sqrt{1+q}+\sqrt{q}} + \sqrt{\sqrt{q}(\sqrt{1+q}+1)}}. \quad (4.18)$$

Proof. When the puncture $a = 1$, we note that $u_1 = u_2 = \frac{\mathcal{K}'}{4}$, and by (122.10) in [5],

$$\operatorname{sn} \frac{\mathcal{K}'}{2} = \frac{1}{\sqrt{1+q}}, \quad \operatorname{cn} \frac{\mathcal{K}'}{2} = \frac{\sqrt{q}}{\sqrt{1+q}}, \quad \operatorname{dn} \frac{\mathcal{K}'}{2} = \sqrt{q},$$

where all Jacobian elliptic function sn , cn and dn have the modulus q' , then the half argument formulas (124.01) in [5] lead to

$$\operatorname{dn} \frac{\mathcal{K}'}{4} = q^{\frac{1}{4}} (\sqrt{1+q}+1)^{\frac{1}{2}} (\sqrt{1+q}+\sqrt{q})^{-\frac{1}{2}}.$$

Combining with (1.5) we obtain (4.18). \square

We can compare the hyperbolic and extremal lengths.

Theorem 4.5 *With the same λ_j as in (1.4), l_j as in (3.16), θ_j as in (3.13), $j = 1, 2$, we have*

$$\frac{l_j}{\pi} \leq \lambda_j \leq \frac{l_j}{\frac{\pi}{2} + \arccos \left(\frac{\sinh(l_j/2)}{\cosh(l_1/2) + \cosh(l_2/2)} \right)}. \quad (4.19)$$

Proof. From [14, (1) and (2)], in the punctured annulus A we have $(\frac{\pi}{2} + \theta_j)\lambda_j \leq l_j \leq \pi\lambda_j$, that is $\frac{l_j}{\pi} \leq \lambda_j \leq \frac{l_j}{\frac{\pi}{2} + \theta_j}$. Substituting (1.3) leads to (4.19). \square

5 Degenerating cases

In this section we consider the degenerating cases when the puncture a is approaching the inner or outer boundary of A and give some connection between the parameter pairs (R, a) and (k, r) , and some other cases when one of the boundaries is shrinking to a point or to the other boundary.

Case (i): R is fixed. When $a \rightarrow R$, by the expression (1.6), we know $u_1 \rightarrow \frac{\mathcal{K}'}{2}$, $u_2 \rightarrow 0$ as $a \rightarrow R$, then $\operatorname{dn} u_1 \rightarrow \sqrt{q}$, $\operatorname{dn} u_2 \rightarrow 1$, and $p_1 \rightarrow 1$, $p_2 \rightarrow \frac{2\sqrt{q}}{q+1}$ (see [5, p. 19]). We note that $\mu(\frac{2\sqrt{q}}{q+1}) = \frac{1}{2}\mu(q) = 2\log R$ (see [1, 5.2]), thus $\lambda_1 \rightarrow +\infty$, $\lambda_2 \rightarrow \frac{\pi}{\log R}$ with λ_1 and λ_2 given by (1.4). From (4.19) we know $l_1 \rightarrow \infty$ for $\lambda_1 \rightarrow \infty$, and thus $k \rightarrow +\infty$ from the first formula in (3.16). When $a \rightarrow 1/R$, it holds $\lambda_1 \rightarrow \frac{\pi}{\log R}$, $\lambda_2 \rightarrow +\infty$. Thus $l_2 \rightarrow \infty$, and $r \rightarrow 1$ from the second formula in (3.16).

Case (ii): a is fixed. When $R \rightarrow +\infty$, A is tending to the thrice-punctured sphere. Since $\mu(q) = 4\log R$, we know $q \rightarrow 0$, and $p_1 \rightarrow 0$, $p_2 \rightarrow 0$ from (1.5). Thus $\lambda_1 \rightarrow 0$, $\lambda_2 \rightarrow 0$, which imply that $l_1 \rightarrow 0$, $l_2 \rightarrow 0$ from (4.19). By the first formula of (3.16), we have $k \rightarrow 1$. The second formula of (3.16) is written as

$$\cosh \frac{l_2}{2} = \frac{1}{k} \frac{1}{r-1} (k^2 - 1 - r + 1) = \frac{1}{k} \left(\frac{k^2 - 1}{r-1} - 1 \right).$$

Thus $\frac{k-1}{r-1} \rightarrow 1$, $k \rightarrow 1$ and $r \rightarrow 1$ when $R \rightarrow +\infty$. In this case from the forms of f and g we see that the covering group G does not converges algebraically, so we need another Fuchsian group to describe the degeneration. Let $h(z) := \frac{k+1}{k-1} \frac{z-1}{z+1}$. We consider $G_1 := hGh^{-1}$. Since $h(1) = 0$, $h(\frac{1}{k}) = -1$, $h(k) = 1$, from the representation of f and g in (2.10), we have

$$hfh^{-1} = \frac{\pm}{2k} \begin{pmatrix} k^2 + 1 & (k+1)^2 \\ (k-1)^2 & k^2 + 1 \end{pmatrix} \rightarrow \pm \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} := f_0, \quad \text{as } R \rightarrow +\infty, \quad (5.20)$$

$$hgh^{-1} = \pm \begin{pmatrix} 1 & 0 \\ 2\frac{(k-1)(r+1)}{(k+1)(r-1)} & 1 \end{pmatrix} \rightarrow \pm \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} := g_0, \quad \text{as } R \rightarrow +\infty. \quad (5.21)$$

That means, when $R \rightarrow +\infty$, h maps the fundamental domain D_A^0 shown in Figure 1(a) onto the domain bounded by $\operatorname{Re} z = \pm 1$ and $\{|z \pm \frac{1}{2}| = \frac{1}{2}\}$. And we know that, as $R \rightarrow +\infty$, G_1 converges algebraically to a Fuchsian group generated by f_0 and g_0 from (5.20) and (5.21). The group $\langle f_0, g_0 \rangle$ is known as the principal congruence subgroup of $PSL(2, \mathbb{Z})$ of level 2 (see e.g. [6, p.54]). This case is the degeneration for the covering group when the two hyperbolic elements g, fg^{-1} are both becoming parabolic ones.

Case (iii): $a = 1$. When $R \rightarrow 1$, we have $q \rightarrow 1^-$ from $\mu(q) = 4 \log R$, thus $p_1 \rightarrow 1^-$, $p_2 \rightarrow 1^-$ and $\lambda_1 \rightarrow +\infty$, $\lambda_2 \rightarrow +\infty$. Therefore $l_1 \rightarrow \infty$, $l_2 \rightarrow \infty$ from (4.19), and $k \rightarrow +\infty$ from the first formula in (3.16). By Corollary 3.3, $k^2 = \frac{3r-1}{3-r}$, we have $r \rightarrow 3$, and $g(z)$ converges to the parabolic transformation $z \mapsto \frac{3z-2}{2z-1}$. This is the degeneration for the covering group when the trace of a hyperbolic element is going to ∞ . In the fundamental domain D_A shown in Figure 1(b), the two semi-circles with end points $\pm \frac{1}{k}$, $\pm k$, are shrinking to two points $0, \infty$, respectively. Then D_A is becoming the triangle $\{z \in \mathbb{H} : |z + \frac{1}{2r}| > \frac{1}{2r} \text{ and } |z - \frac{r}{2}| > \frac{r}{2}\}$. In this case, A is conformally equivalent to the once-punctured unit disk.

Case (iv): a/R is fixed. Now we let $R \rightarrow +\infty$ and take the once-punctured annulus model as $A_1 = \{R^{-2} < |z| < 1\} \setminus \{x\}$, $x = a/R$, $R^{-2} < x < 1$. First we consider $\lim_{R \rightarrow +\infty} \operatorname{sn}(v, q')$, where $v := \frac{2\mathcal{K}}{\pi} \log x$, $\mathcal{K} = K(q)$ and q satisfies $\mu(q) = 4 \log R$. All the Jacobian functions sn , cn , dn in this section have the modulus q' , we will omit q' in the notation of them. Since

$$\int_0^{\operatorname{sn} v} \frac{dt}{\sqrt{(1-t^2)(1-(1-q^2)t^2)}} = v, \quad (5.22)$$

and $q \rightarrow 0$, $\mathcal{K} \rightarrow \frac{\pi}{2}$ as $R \rightarrow +\infty$, taking limits of both sides of (5.22) gives

$$\lim_{R \rightarrow +\infty} \operatorname{sn} v = \frac{x^2 - 1}{x^2 + 1}, \quad (5.23)$$

and thus

$$\lim_{R \rightarrow +\infty} \operatorname{cn} v = \frac{2x}{x^2 + 1}, \quad \lim_{R \rightarrow +\infty} \operatorname{dn} v = \frac{2x}{x^2 + 1}. \quad (5.24)$$

Note that

$$\operatorname{dn} u_1 = \operatorname{dn} \left(\frac{2\mathcal{K}}{\pi} \log R^2 x \right) = \operatorname{dn} \left(\frac{\mathcal{K}'}{2} + v \right) = \frac{\sqrt{q} (\operatorname{dn} v - (1-q) \operatorname{sn} v \operatorname{cn} v)}{1 - (1-q) \operatorname{sn}^2 v},$$

where the last equivalence is due to [5, 122.10 and 123.01]. Then from (5.23) and (5.24),

$$\lim_{R \rightarrow +\infty} \frac{\operatorname{dn} u_1}{\sqrt{q}} = \lim_{R \rightarrow +\infty} \frac{\operatorname{dn} v - (1-q) \operatorname{sn} v \operatorname{cn} v}{1 - (1-q) \operatorname{sn}^2 v} = \frac{1}{x}.$$

Hence for p_1 as in (1.5), we have

$$\lim_{R \rightarrow +\infty} p_1 = \lim_{R \rightarrow +\infty} \frac{\operatorname{dn} u_1 + 1}{\sqrt{q} + \operatorname{dn} u_1 / \sqrt{q}} = \lim_{R \rightarrow +\infty} \frac{\sqrt{q}}{\operatorname{dn} u_1} = x. \quad (5.25)$$

Therefore $\lambda_1 \rightarrow \frac{2\pi}{\mu(x)}$. By the second formula in (1.6) we see u_2 is fixed, so $\operatorname{dn} u_2$ is fixed to be a finite number. Thus p_2 in (1.5) is tending to 0, and $\lambda_2 \rightarrow 0$ as $R \rightarrow +\infty$. In this case, $\lim A_1$ is conformally equivalent to the twice-punctured unit disk $\mathbb{D} \setminus \{0, x\}$, and $\lim_{R \rightarrow +\infty} \lambda_1$ coincides with $\lambda(\tilde{\mathcal{C}})$ in (4.17). To show the degeneration of A_1 in terms of the deformation of the covering group G , we let $h_1(z) := \frac{r+1}{r} \frac{z-r}{z-1}$ and consider $h_1 G h_1^{-1}$. Then $h_1(1) = \infty$, $h_1(\frac{1}{r}) = \frac{(r+1)^2}{r}$, $h_1(r) = 0$. And $h_1 f h_1^{-1}$, $h_1 g h_1^{-1}$, $h_1 f g^{-1} h_1^{-1}$ satisfy

$$h_1 f h_1^{-1} = \frac{\pm 1}{k(r^2 - 1)} \begin{pmatrix} -(r+1)(k^2 - r) & (r+1)^2(k^2 - 1) \\ -r(k^2 - 1) & (r+1)(k^2 r - 1) \end{pmatrix} \rightarrow \pm \begin{pmatrix} -1 & \frac{(r+1)^2}{r} \\ -1 & \frac{(r+1)^2}{r} - 1 \end{pmatrix}$$

as $R \rightarrow +\infty$, and

$$h_1 g h_1^{-1} = \pm \begin{pmatrix} 1 & -\frac{(r+1)^2}{r} \\ 0 & 1 \end{pmatrix}, \quad (5.26)$$

$$h_1 f g^{-1} h_1^{-1} = \frac{\pm 1}{k(r^2 - 1)} \begin{pmatrix} -(r+1)(k^2 - r) & (r+1)^2(r - \frac{k}{r}) \\ -r(k^2 - 1) & (r+1)(r - k^2) \end{pmatrix} \rightarrow \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

as $R \rightarrow +\infty$. Thus, $l_1 \rightarrow 2 \log r$, $l_2 \rightarrow 0$. In such a way our once-punctured annulus model A_1 becomes the twice-punctured unit disk discussed in Beardon's paper [3]. And $t = -\frac{(r+1)^2}{r}$, for t as in [3, (5)], r as in (5.26). We note that in this case $\cos \theta_1 \rightarrow \tanh \frac{l_1}{4}$ from the first expression of (1.3), then $\tan \theta_1 \rightarrow 1 / \sinh \frac{l_1}{4}$, and for $j = 1$, (4.19) becomes

$$\frac{l_1}{\pi} \leq \lambda_1 \leq \frac{l_1}{\frac{\pi}{2} + \arctan \left(\frac{1}{\sinh(l_1/4)} \right)},$$

which is the same as Theorem 8.2 in [3].

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